Problem 1.55

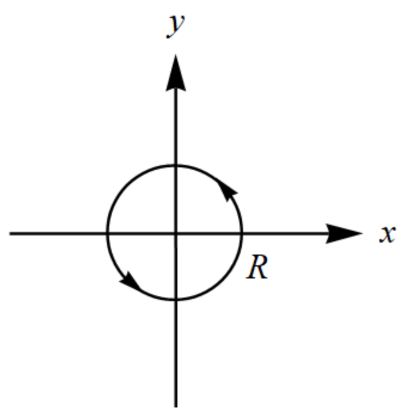
Check Stokes' theorem using the function $\mathbf{v} = ay\hat{\mathbf{x}} + bx\hat{\mathbf{y}}$ (a and b are constants) and the circular path of radius R, centered at the origin in the xy plane. [*Answer:* $\pi R^2(b-a)$.]

Solution

Stokes's theorem relates the integral of a curl over an open surface to a closed loop integral over that surface's boundary line.

$$\iint_S (\nabla \times \mathbf{v}) \cdot d\mathbf{S} = \oint_{\text{bdy } S} \mathbf{v} \cdot d\mathbf{l}$$

Let S be a positively oriented disk in the xy-plane with radius R, centered at the origin.



Parameterize the boundary line.

$$\mathbf{l}(t) = \langle R\cos t, R\sin t, 0\rangle, \quad 0 \leq t \leq 2\pi$$

Evaluate the left side, noting that the direction of the area element is given by the right-hand corkscrew rule.

$$\begin{aligned}
\iint_S (\nabla \times \mathbf{v}) \cdot d\mathbf{S} &= \iint_S \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ay & bx & 0 \end{vmatrix} \cdot d\mathbf{S} \\
&= \iint_S \left\{ \hat{\mathbf{x}} \left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(bx) \right] - \hat{\mathbf{y}} \left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(ay) \right] + \hat{\mathbf{z}} \left[\frac{\partial}{\partial x}(bx) - \frac{\partial}{\partial y}(ay) \right] \right\} \cdot d\mathbf{S} \\
&= \iint_S \{ \hat{\mathbf{x}}[(0)-(0)] - \hat{\mathbf{y}}[(0)-(0)] + \hat{\mathbf{z}}[(b)-(a)] \} \cdot d\mathbf{S} \\
&= \int_0^{2\pi} \int_0^R (b-a)\hat{\mathbf{z}} \cdot (\hat{\mathbf{z}} r\, dr\, d\phi) \\
&= (b-a) \left(\int_0^R r\, dr \right) \left(\int_0^{2\pi} d\phi \right) = (b-a)\left(\frac{R^2}{2}\right)(2\pi) = \pi R^2 (b-a)
\end{aligned}$$

Evaluate the right side.

$$
\begin{aligned}
\oint_{\text{bdy } S} \mathbf{v} \cdot d\mathbf{l} &= \int_0^{2\pi} \mathbf{v}(\mathbf{l}(t)) \cdot \mathbf{l}'(t)\,dt \\
&= \int_0^{2\pi} \langle a(R\sin t), b(R\cos t), 0\rangle \cdot \langle -R\sin t, R\cos t, 0\rangle\,dt \\
&= \int_0^{2\pi} (-aR^2\sin^2 t + bR^2\cos^2 t)\,dt \\
&= R^2 \int_0^{2\pi} (b\cos^2 t - a\sin^2 t)\,dt \\
&= R^2 \left(b\int_0^{2\pi} \cos^2 t\,dt - a\int_0^{2\pi} \sin^2 t\,dt \right) \\
&= R^2 \left[b\int_0^{2\pi} \frac{1}{2}(1+\cos 2t)\,dt - a\int_0^{2\pi} \frac{1}{2}(1-\cos 2t)\,dt \right] \\
&= \frac{R^2}{2}\left[b\left(t + \frac{1}{2}\sin 2t\right)\Big|_0^{2\pi} - a\left(t - \frac{1}{2}\sin 2t\right)\Big|_0^{2\pi} \right] \\
&= \frac{R^2}{2}[b(2\pi) - a(2\pi)] \\
&= \pi R^2 (b - a)
\end{aligned}
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